HOMOMORPHISMS FROM A FINITE GROUP INTO WREATH PRODUCTS

JAN-CHRISTOPH SCHLAGE-PUCHTA

Abstract. Let G be a finite group, A a finite abelian group. Each homomorphism $\varphi : G \to A \wr S_n$ induces a homomorphism $\overline{\varphi} : G \to A$ in a natural way. We show that as φ is chosen randomly, then the distribution of $\overline{\varphi}$ is close to uniform. As application we prove a conjecture of T. Müller on the number of homomorphisms from a finite group into Weyl groups of type D_n.

Let G be a finite group, A a finite abelian group. In this article we consider the number of homomorphisms $G \to A \wr S_n$, where n tends to infinity. These numbers are of interest, since they encode information on the isomorphism types of subgroups of index n, confer [2], [3]. If $\varphi : G \to A \wr S_n$ is a homomorphism, we can construct a homomorphism $\overline{\varphi} : G \to A$ as follows. We represent the element $\varphi(g) \in A \wr S_n$ as $(\sigma; a_1, \dots, a_n)$, where $\sigma \in S_n$ and $a_i \in A$, and then define $\overline{\varphi}(g) = \prod_{i=1}^n a_i$. The fact that $\overline{\varphi}$ is a homomorphism follows from the fact that A is abelian and the definition of the product within a wreath product. In this article we prove the following.

Theorem 1. *Let G be a finite group of order d, A a finite abelian group. Define the distribution function δ_n on $\mathrm{Hom}(G, A)$ as the image of the uniform distribution on $\mathrm{Hom}(G, A \wr S_n)$ under the map $\varphi \mapsto \overline{\varphi}$. Then there exist positive constants c, C, independent of n, such that $\|\delta_n - u\|_\infty < Ce^{-cn^{1/d}}$, where u is the uniform distribution, and $\|\cdot\|_\infty$ denotes the supremum norm.*

As an application we prove the following, which confirms a conjecture by T. Müller.

Corollary 2. *For a finite group G there exists a constant $c > 0$, such that if W_n denotes the Weyl group of type D_n, then*

$$|\mathrm{Hom}(G, W_n)| = \left(\frac{1}{1 + s_2(G)} + \mathcal{O}(e^{-cn^{1/d}})\right) |\mathrm{Hom}(G, C_2 \wr S_n)|$$

This assertion was proven by T. Müller under the assumption that G is cyclic (confer [1, Proposition 3]). Different from his approach we do not enumerate homomorphisms φ with given image $\overline{\varphi}$, but directly work with the distribution of $\overline{\varphi}$, that is, we obtain the relation between $|\mathrm{Hom}(G, W_n)|$ and $|\mathrm{Hom}(G, C_2 \wr S_n)|$ without actually computing these functions.

Denote by $\pi : A \wr S_n \to S_n$ the canonical projection onto the active group. The idea of the proof is to stratify the set $|\mathrm{Hom}(G, A \wr S_n)|$ according to $\pi \circ \varphi \in \mathrm{Hom}(G, S_n)$. It turns out that in strata such that $\pi \circ \varphi(G)$ viewed as a permutation group on $\{1, \dots, n\}$ has a fixed point the distribution of $\overline{\varphi}$ is actually uniform, while the probability of having no fixed point is very small.

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Lemma 3. *Let $\sigma : G \to S_n$ be a homomorphism such that $\sigma(G)$ has a fixed point. Define the set*

$$M = \{\varphi : G \to A \wr S_n : \pi \circ \varphi = \sigma\}.$$

Then the function $M \to \mathrm{Hom}(G, A)$ mapping φ to $\overline{\varphi}$ is surjective, and all fibres have the same cardinality.

Proof. Without loss we may assume that the point n is fixed. Let σ_1 be the restriction of σ to the set $\{1, \dots, n-1\}$. Then

$$M = \{\varphi_1 : G \to A \wr S_{n-1} : \pi \circ \sigma = \sigma_1\} \times \mathrm{Hom}(G, A),$$

hence, for each $\psi : G \to A$ and each $\varphi_1 : G \to A \wr S_{n-1}$ with $\pi \circ \varphi_1 = \sigma_1$ there is precisely one $\varphi \in M$ with $\overline{\varphi} = \psi$ which coincides with φ_1 on $A \wr S_{n-1}$. This implies that all fibres have the same cardinality. Defining $\varphi : G \to A \wr S_n$ by $\varphi(g) = (\sigma(g), 1, \dots, 1)$ we see that M is non-empty, which implies the surjectivity. □

To bound the number of homomorphisms φ for which $\pi \circ \varphi$ has no fixed point we need the following, which is contained in [2, Proposition 1], in particular the equality of equations (8) and (9) in that article.

Lemma 4. *Let G be a group, A a finite abelian group, $U \leq G$ a subgroup of index k, $\varphi_1 : G \to S_k$ the permutation representation given by the action of G on G/U. Then the number of homomorphisms $\varphi : G \to A \wr S_k$ with $\pi \circ \varphi = \varphi_1$ equals $|A|^{k-1} |\mathrm{Hom}(U, A)|$.*

We use this to prove the following.

Lemma 5. *Let G be a group of order d, A a finite abelian group, $\varphi : G \to A \wr S_n$ be a homomorphism chosen at random with respect to the uniform distribution. Then there is a constant $c > 0$, depending only on G, such that the probability that $\pi \circ \varphi(G)$ has no fixed points is $\mathcal{O}(e^{-cn^{1/d}})$.*

Proof. Let $U_1, \dots, U_\ell$ be a complete list of subgroups of G up to conjugation, where $U_\ell = G$. To determine a homomorphism $\varphi : G \to A \wr S_n$ we first have to choose a homomorphism $\sigma : G \to S_n$, and then count the number of ways in which this homomorphism can be extended to a homomorphism into $A \wr S_n$. Suppose that the action of G on $\{1, \dots, n\}$ induced by σ has m_i orbits on which G acts similar to the action of G on G/U_i. Then by the previous lemma we find that there are

$$\prod_{i=1}^{\ell} \left(|A|^{(G:U_i)-1} |\mathrm{Hom}(U_i, A)|\right)^{m_i}$$

possibilities to extend σ. Next we compute the number of ways σ can be chosen such that σ realizes given values $m_1, \dots, m_\ell$. Choices of σ correspond to subgroups of S_n conjugate to some fixed subgroup with the given number of orbits of the respective types, and the number of such subgroups is $(S_n : C_{S_n}(\sigma(G))$. We have $C_{S_n}(\sigma(G)) = \times_{i=1}^{\ell} C_{\mathrm{Sym}(G/U_i)}(G) \wr S_{m_i}$, hence, defining $c_i = |C_{\mathrm{Sym}(G/U_i)}(G)|$ we find that σ can be chosen in $\frac{n!}{\prod_{i=1}^{\ell} m_i! c_i^{m_i}}$ different ways. Combining these results we obtain

$$|\mathrm{Hom}(G, A \wr S_n)| = n! \sum_{\substack{m_1, \dots, m_\ell \\ m_1 + \dots + m_\ell = n}} \prod_{i=1}^{\ell} \frac{\left(|A|^{(G:U_i)-1} |\mathrm{Hom}(U_i, A)|\right)^{m_i}}{m_i! c_i^{m_i}}.$$

We claim that terms with $m_\ell = 0$ are small when compared to the whole sum. Since the number of summands is polynomial in n, it suffices to show that for every tuple $(m_1, \dots, m_{\ell-1}, 0)$ there exists a tuple $(m_1', \dots, m_{\ell-1}', m_\ell')$ with $m_\ell' \neq 0$, such that the summand corresponding to the first tuple is smaller than the one corresponding to the second by a factor $e^{cn^{1/d}}$. We do so by explicitly constructing the second tuple. Without loss we may assume that in the first tuple m_1 is maximal. We then set $m_1' = m_1 - \lfloor cn^{1/d} \rfloor$, $m_\ell' = (G : U_1)\lfloor cn^{1/d}\rfloor$, and $m_i' = m_i$ for $i \neq 1, \ell$, where c is a positive constant chosen later. Then the product on the right hand side of the last displayed equation changes by a factor

$$\frac{m_1!}{(m_1 - \lfloor cn^{1/d}\rfloor)!}\left(\frac{|A|^{(G:U_1)-1}|\operatorname{Hom}(U_1, A)|}{c_1|\operatorname{Hom}(G, A)|^{(G:U_1)}}\right)^{-\lfloor cn^{1/d}\rfloor}\frac{1}{\big((G:U_1)\lfloor cn^{1/d}\rfloor\big)!}.$$

We may assume that n is sufficiently large, so that $m_1 > 2\lfloor cn^{1/d}\rfloor$. We can then estimate the factorials using the largest and smallest factors occurring. The other terms can be bounded rather careless to find that this quotient is at least

$$\left(\frac{m_1}{\big(cdn^{1/d}|A|\big)^d|\operatorname{Hom}(U_1, A)|}\right)^{\lfloor cn^{1/d}\rfloor}.$$

Since m_1 was chosen maximal we have $m_1 \geq \frac{n}{|G|\ell}$, and we find that for $c^{-1} = ed\ell|A||\operatorname{Hom}(U_1, A)|$ the last expression is at least $e^{\lfloor cn^{1/d}\rfloor}$. Since c depends only on the subgroup U_1, we can take the minimum value over all the finitely many subgroups and obtain that there exists an absolute constant $c > 0$, such that the number of homomorphisms φ such that $\pi \circ \varphi$ has no fixed point is smaller by a factor $\mathcal{O}(e^{-cn^{1/d}})$ than the number of all homomorphisms. □

To prove the theorem let $\varphi : G \to A \wr S_n$ be chosen with respect to the uniform distribution. Let p be the probability that $(\pi \circ \varphi)(G)$ has no fixed point. By Lemma 3 we see that the conditional distribution of $\overline{\varphi}$ subject to the condition that $(\pi \circ \varphi)(G)$ has a fixed point is uniform, hence $\delta = (1 - p)u + p\delta^0$ for some distribution function δ^0. This implies $\|\delta - u\|_\infty \leq p$. By Lemma 5 we see that $p = \mathcal{O}(e^{-cn^{1/d}})$, and our claim follows.

To deduce the corollary note that W_n is the subgroup of $C_2 \wr S_n$ defined by the condition $(\pi; a_1, \dots, a_n) \in W_n \Leftrightarrow a_1 \cdots a_n = 1$, that is, a homomorphism $\varphi : G \to C_2 \wr S_n$ has image in W_n if and only if $\overline{\varphi} : G \to C_2$ is trivial. By the theorem the probability for this event differs from the probability that a random homomorphism $G \to C_2$ is trivial by $\mathcal{O}(e^{-cn^{1/d}})$, hence, we have

$$|\operatorname{Hom}(G, W_n)| = \left(\frac{1}{|\operatorname{Hom}(G, C_2)|} + \mathcal{O}(e^{-cn^{1/d}})\right)|\operatorname{Hom}(G, C_2 \wr S_n)|.$$

But there is a bijection between non-trivial homomorphisms $G \to C_2$ and subgroups of index 2, hence, $|\operatorname{Hom}(G, C_2)| = 1 + s_2(G)$, and the corollary follows.

References

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Krijgslaan 281, Gebouw S22, 9000 Gent, Belgium
E-mail address: `jcsp@cage.ugent.be`